Z. Sylvan – 24/2/11 – SLagrangian & SYZ (II)

- Last time:

(X,Ω) Calabi-Yau (Kähler, $\Omega \in \Omega^{n,0}(X)$ nonvanishing)

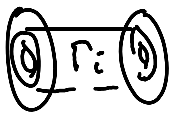
$L_0 \subset X$ special Lagrangian ($\mathrm{Im}\, \Omega_{|L} = 0$)

$B = \{SL \text{ deform}^{ns} \text{ of } L\}$ was a real mfld of $\dim_{\mathbb{R}} = b^1(L)$ with 2 affine structures:

γ_i: basis of $H_1(L_0)$

Γ_i = cylinder swept by γ_i under deformation $\rightsquigarrow x_i(L) = \int_{\Gamma_i} \omega$ affine coords.

$$dx_i(v) = \int_{\gamma_i} \iota_v \omega$$



Similarly γ_i^* basis of $H_{n-1}(L_0) \rightsquigarrow x_i^*(L) = \int_{\Gamma_i^*} \mathrm{Im}\,\Omega$

Γ_i^*

$$dx_i^*(L) = \int_{\gamma_i^*} \iota_v \mathrm{Im}\,\Omega.$$

[We assume $L_0 \simeq T^n$].

Now, let $M = \{(L,\nabla) / L \in B,\ \nabla \text{ flat } U(1) \text{ connection on } L \times \mathbb{C}\} / \text{gauge equiv}^{ce}$

($\nabla = d + iA$, A closed 1-form on L, mod exact 1-forms & d of S^1-valued functions).

- We can think of ∇ as a map $H_1(L) \to U(1)$.
- Pick harmonic representative of A as our favorite one.

Goal: show M is naturally a Calabi-Yau mfld.

- $$T_{(L,\nabla)} M = \{(v, i\alpha) \in C^\infty(NL) \oplus i\mathcal{H}^1(L) / -\iota_v\omega \in \mathcal{H}^1(L)\} \cong \mathcal{H}^1(L;\mathbb{C})$$
$$(v,\alpha) \longmapsto -\iota_v\omega + \frac{i\alpha}{2\pi}$$

complex vector space

$J^\vee \in \mathrm{End}(T_{(L,\nabla)}M)$ the natural almost-$\mathbb{C}$ structure wrt this isom.

i.e. $$(0, i\alpha) \mapsto \left(\tfrac{1}{2\pi}\omega^{-1}(\alpha), 0\right)$$
$$(v,0) \mapsto (0, -2\pi i\, \iota_v\omega).$$

Claim: $J^\vee$ is integrable.

__Pf:__ near (L_0, ∇_0), $z_i(L,\nabla) := \frac{1}{2\pi}\int_{\gamma_i} i(A-A_0) - \int_{\Gamma_i} \omega \; := \frac{i}{2\pi}\theta_i - x_i$

is a local complex coordinate:

$$dz_i(v, i\alpha) = \int_{\gamma_i} \frac{i\alpha}{2\pi} - \int_{\gamma_i} \iota_v \omega = \left\langle \left[-\iota_v\omega + \frac{i\alpha}{2\pi}\right], \gamma_i \right\rangle$$

$\mathbb{C}$-linear form on $\mathcal{H}^1(L,\mathbb{C})$ ✓

or directly: $dz_i(J^\vee(v,i\alpha)) = i\, dz_i(v, i\alpha)$.

__Claim 2:__ $\|$ M is Calabi-Yau.

- Def. $\Omega^\vee = c.\, dz_1 \wedge \dots \wedge dz_n$ in above local coordinates. Intrinsically:

$$\Omega^\vee((v_1, i\alpha_1), .., (v_n, i\alpha_n)) = c.\int_L \left(-\iota_{v_1}\omega + \frac{i\alpha_1}{2\pi}\right) \wedge \dots \wedge \left(-\iota_{v_n}\omega + \frac{i\alpha_n}{2\pi}\right)$$

- Let $\omega^\vee((v_1, i\alpha_1), (v_2, i\alpha_2)) = \int_L \left(\frac{\alpha_2}{2\pi} \wedge \iota_{v_1} \operatorname{Im}\Omega - \frac{\alpha_1}{2\pi} \wedge \iota_{v_2} \operatorname{Im}\Omega\right)$

(recall: we've normalized $\int_L \Omega = 1$).

$\omega^\vee$ is closed: indeed $\omega^\vee = \sum dx_i^* \wedge d\theta_i$ (because $\gamma_i \leftrightarrow \gamma_i^*$ dual
& nondegenerate basis of homology).

- $\omega^\vee$ is $J^\vee$-compatible:

$$\omega^\vee((v_1, i\alpha_1), J^\vee(v_2, i\alpha_2)) = \int_L -\iota_{v_2}\omega \wedge \iota_{v_1}\operatorname{Im}\Omega - \frac{\alpha_1}{2\pi}\,\iota_{\omega^{-1}\alpha_2}\operatorname{Im}\Omega$$

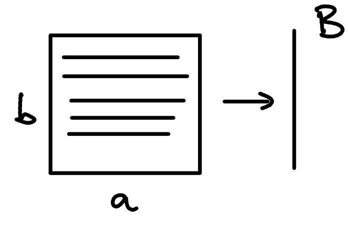
$$= \int_L (-\iota_{v_2}\omega) \wedge *(-\iota_{v_1}\omega) + \int_L \frac{-\alpha_1}{2\pi} \wedge *\left(\frac{-\alpha_2}{2\pi}\right)$$

$$= \int_L \langle \iota_{v_2}\omega, \iota_{v_1}\omega\rangle + \int_L \left\langle \frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi}\right\rangle \quad \text{pos. def. symmetric ✓}$$

__Example:__ $X = T^2 = \mathbb{C}/a\mathbb{Z} + ib\mathbb{Z}$, $\omega = \frac{i}{2} dz \wedge d\bar{z}$ standard symplectic form
standard J

$\Omega = \frac{1}{a} dz$

(horizontal S^1's are SLag, of $\int_L \Omega = 1$)

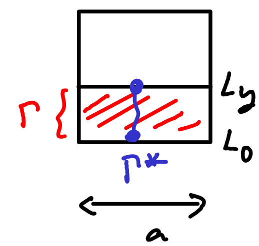


- $B = i\mathbb{R}/ib\mathbb{Z}$
- A connection on $L_y = \{\operatorname{Im} z = y\}$ $\Longleftrightarrow$ 1-form $d + iA\,dx$, $A = $ const.

up to gauge, $A \in \mathbb{R}/\frac{2\pi}{a}\mathbb{Z}$

(because $g = e^{\frac{2\pi}{a} ix} : L_y \to S^1$ gauge transf. w/ $g^{-1}dg = \frac{2\pi}{a} i\,dx$.)

(or: parallel transport holonomy is $e^{-iAa} \in S^1$, conn/gauge $\leftrightarrow$ holonomy)



$$\rightsquigarrow \quad z^\vee(L_y, iA\,dx) = -\int_\Gamma \omega + \int_\gamma \frac{iA}{2\pi}$$
$$= -ay + ia\frac{A}{2\pi}$$

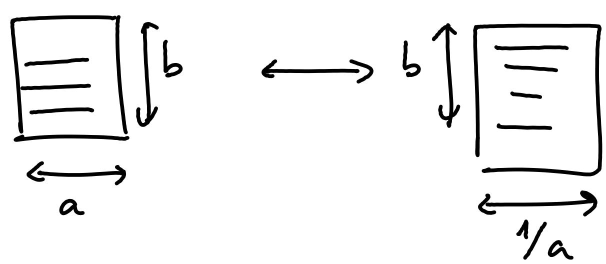
$$dz^\vee(u\partial_y, iv\,dx) = -au + ia\frac{v}{2\pi}, \qquad \Omega^\vee = -i\,dz^\vee$$

and $x^\vee(L_y, iA\,dx) = \int_{\Gamma^*} \operatorname{Im}\Omega = \frac{y}{a}$ $\quad\Rightarrow\quad \omega^\vee = \frac{1}{a}\,du \wedge dv$

... Point: M looks like $\mathbb{C}/\frac{1}{a}\mathbb{Z} + ib\mathbb{Z}$, ω_{std}, J_{std}

(with coordinate $\frac{-iz^\vee}{a}$)

i.e.



$\int \omega = ab$ $\quad\longleftrightarrow\quad$ $\tau = ab$

$\tau = \frac{b}{a}$ $\quad\longleftrightarrow\quad$ $\int \omega = \frac{b}{a}$